STOCHASTIC PROGRAMMING

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BY

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TO WHOM IT MAY CONCERN

This is to certify that Mr. Ghulam Haider has completed his dissertation on "STOCHASTIC PROGRAMMING" under my supervision. The work is original and is suitable for submission for the award of M.Phil degree in Statistics.

(Dr. M.M. Khalid)

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P R E F A C E

This dissertation entitled 'STOCHASTIC PROGRAMMING' is devoted to the methods of solution and applications of various probilistic models.

It consist of four chapters. In chapter I, which is introductory character, consideration is given to the problems of risk and uncertainty, used for various modelling systems based on two-stage, chance constrained and some theoretical approaches. Chapter II, deals with the two-stage stochastic programming techniques in various models with qualitative, analysis and their applications, Chapter III, is devoted to giving a brief sketch of chance - constrained programming, varigus model, technique for numerical example and their applications in different field. Chapter IV is the last chapter in which we have presented different applications of stochastic programming model., viz, trasnportation problem, human resources pnanning and commerical business,

I express my obligation to my superrisor Dr. M.M. Khalid, who had been a source of inspiration and constant encouragement, throughout the completion of this work. He supervised the work with profound interest. It is a great pleasure to me to express my extreme gratefulness to

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Chapter - I

INTRODUCTION (STOCHASTIC PROGRAMMING)

Chapter - I

MATHEMATICAL PROGRAMMING

Mathematical programming is concerned with finding solution to the problems of decision making under limited resources to meet the desired objectives.

The mathematical programming problem (MPP) can be formulated as :

Maximize (or Minimize) $Z = f\ (X)$

Subject to the constraints $g_i(X)\ [\ \leq\ ,\ =\ \geq\ ]\ b_i$

$$i = 1, 2, - m \qquad (1.1.1)$$

and non-negative restrictions $X \geq 0$.

Where $\underline{X}' = (X_1, X_2 - X_n)$ is an n-component vector of variables, $f(x)$ and $g_i(X)$ are functions of n-variables $X_1, - X_n$, b_i are known constants. Furthermore one and only one of the signs $\leq$, $=$ and $\geq$ holds for each constraints,

Depending upon the nature of the objective function $f(\underline{X})$ the functions $gi(X)$ in the constraints and other restrictions on the variables.

Vector X the MPP may be classified under different headings. Although no technique has been found to be universally applicable for every class of MPP.

Separate algorithms are available for all most all classes, some important classes are listed below :

1. Linear programming
2. Non-linear programming
3. Quadratic programming
4. Dynamic Programming
5. Integer Programming
6. Stochastic Programming
7. Goal Programming
8. Parametric Programming
9. Chance constrained Programming
10. Geometric Programming
11. Separable programming

1.2 BRIEF HISTORICAL SKETCH OF MATHEMATICAL PROGRAMMING

The concept of an optimal solution is very old but the term 'optimum' have been coined by the mathematician leibniz in 1710. Newton and leibniz by means of calculus proved that for a class of problems an optimum solu exists and developed the method for finding it. On the trails of the above two significant. contributions were made to the optimization theory to deal with the restriction or 'constraints'. The technique of optimization for constrained problem by the use of unknown multipliers become known by the name of its inventor Lagrauge as lagrange multipliers technique.

The development of simplex method for linear programming problem by Dauzig in 1947 and work by Khun and Tucker in 1951 on the necessary and sufficient conditions for optimal solu. of a programming problem laid the foundations of developlments in MPP. The contributions of Zoutendgik and Resen to non-linear programming (NLP) during the first half of 1960's have been valuable. Geometric progzamming was developed in 1960's by Duffin, Zener and Peterson. Gomovy told the pioneering work in integer programming Dantzig. Charues and cooper developed stochastic programming techniques.

1.3 STOCHASTIC PROGRAMMING

A linear programming problem is said to be stochastic if one or more of the coefficients in the objective function or the system of constraints or recource availabilities is known only by its probability distribution. Various approaches are available in this case, which may be classified in to three broad types. Chance constrained programming, two stage programming under uncertainty and stochastic linear programming. For problems of stochastic linear programming a distinction is usually made between two related approaches to stochastic programming, the passive and active approach respectively. In the passive approach to stochastic linear programming the statistical distribution of the optimum value of the objective function is estimated either exactly or approximately by numerical method and optimal decision rules are based on the different characteristics of the estimated distribution. In the active approach a new set of decision variables are introduced which indicate the proportions of different resources to be allocated for the various activities, one effect of introducing this set of new decision variables in the active approach is the truncation of the statistical distribution of optimal value of the objective function of the passive approach.

An ordinary linear programming problem is formulated as follows :

Maximize $Z = C'x$ 1.3.1

Under the conditions :

$Ax \leqq b$ 1.3.2

$x \geqq 0$ 1.3.3

Here x and C are column vectors with n elements, A is a matrix with m rows and n columns, b is a column vector within elements and prime denoted x transposition. The maximization is performed have with respect to the elements of the vector x. This is, of course, also a dual minimization problem. Various approaches are available in this case, which may be classified into three broad types.

(a) Chance constrained programming [10]. Here we replace condition (1.3.2) by

$$P\ (A_x \leq b)\ \geq \alpha \tag{1.3.4}$$

Where P stands for probability and α is a column vector with m element with $0 \leq \alpha_i \leq 1$, $i = 1,2,$ — m. The vector α contains a prescribed set of constants that are probability measures of the extent to which constraint violations are admitted.

(b) Two-stage programming under uncertainty [33]. This may be briefly formulated as follows :

$$Z = C'x + E\ \min_y f'y = \text{minimum} \tag{1.3.5}$$

$$A_x + B_y \geq b \tag{1.3.6}$$

$$x \geq 0,\ y \geq 0 \tag{1.3.7}$$

Here E means mathematical expectation f and y are column vectors with r elements and B is a matrix with m rows and r columns. The minimization is performed with respect to the elements of the vectors x and y and the elements of the vector b are assumed to be random variables with known distribution.

(c) Stocheastic Linear programming [51].

A somewhat different line of approach to stochastic linear programming. First suggested by Titner [51] is concerned with the specification of the statistical distribution of the objective function and their implications for decision making under risk. Here we make the assumption that in the linear programming problem defined in the begining in equations (1.3.1) (1.3.2) and (1.3.3) the elements of the vectors b, c and a matrix. A are random variables with a known probability distribution say,

$$P\ (A,\ b,\ c) \qquad \ldots (1.3.8)$$

We distinguish here between two approaches termed by Vajda[52] distribution problems and expected value problems. In the expected value problem studied by Dantzig [16] and others[32] we conider the optimization of the expected value of the objective function. In the distribution problem we try to derive the statistical distribution of the objective function Z. With the distribution problem as studied by Titner, We have two approaches :

(a) the passive and (b) the active approach .

In the passive approach we derive (by numerical method, if necessary) the distribution of the objective function Z under the assumption of a known probability distribution (1.3.8) of all the parameters of the problem. This approach assumes that in almost all possible situations, ie. for almost all possible variations of the parameters, the conditions of the simple non-stochastic linear program are fulfilled and the maximum achieved.

Now consider the active approach. This is the following problem :

$$\text{Maximize} \quad Z = C'x \qquad \ldots.(1.3.9)$$

under the conditions

$$AX \leqq B\,U \qquad \ldots.(1.3.10)$$

$$x \geqq 0$$

Here U is a square matrix with n rows with elements U_{ij} and

$$U_{ij} \geqq 0 \qquad \sum_{j=1}^{n} u_{ij} = 1 \qquad \ldots.(1.3.12)$$

Further, x is a square, diagonal matrix with elements of the vector x in the diagonal and B is a square, diagonal matrix with the elements of the vector C in the diagonal.

Again the problem is the derivation of the probability distribution of maxz , given the probability distribution (1.3.8). But now the probability distribution of the optimal objective function will depend upon the allocation matrix $U = [u_{ij}]$. Now we have only one production situation. The entrepreneur will consider the probability distributions generated by various allocations of resources u_{ij}. Or in our second example we consider the problem of economic planning. Then the proportions

u_{ij} may be allocated to various industries and the central planner will consider the probability distributions of the maximal objective function Z generated by these allocations.

In the first place, interpreting the active approach as a policy model, the elements u_{ij} of the allocation matrix U may be considered as instrument variables in Tinbergen's sense [49], which may be appropriately chosen to optimize a risk preference unctional associated with the objective function. Let Za denote the value of the objective function under active approach and let U and U represent two different sets of resource allocations that could be selected by the policy-maker. Since, in every case, all resources are to be fully allocated by condition (1.3.12), the selections U and U represent only different relative allocations for every resource $i = 1,2, ---,n$. The resulting probability distributions for 'maxza' induced by these two selections are illustrated in the following diagram.1.3

This diagram and interpretation.

F = Probability of getting a value of Z or less.

Diagram

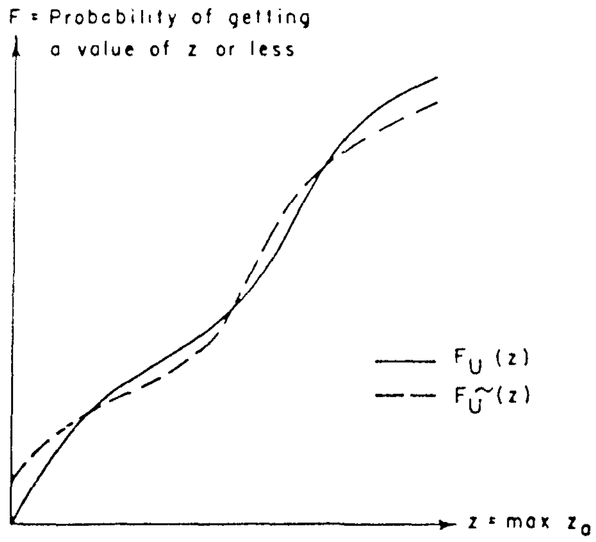

Fig 1.3

Since under stochastic ~~ux~~ linear programming it is assumed that a utility function (i.e. a risk preference function) is available which permits an ordering between all pairs like Fu and $F\tilde{u}$, on this assumption the problem is formally solved at the 'policy level' when the probability distributions are available for each addmissible U. In principle, different decision rules could be compared. For instance, if the utility function is such that only linear decision rules by Theil[50] which depend by definition on the first moment of the cummulative distributions FU and $F\tilde{U}$, are considered by the policy-maker than the allocation policy U may be said to 'better' than U so long as first moment (i.e., expected value) of FU exceeds $F\tilde{u}$. However, even in this case of linear decision rules, an optimum choice problem exists between alternative location parameters characterizing the probability distributions FU and $F\tilde{U}$ and hence, between alternative policies U and $\tilde{U}$.

We may also note that the introduction of additional decision constraints by means of the decision variables u_{ij} has the effect of truncating the statistical distribution [41] of the optimal value of the objective function of the passive approach. In other words, let us assume that a particular allocation matrix U is chosen at first on an a priory basis, then with further observations of the data and the result of more complete specification of probability distribution, another allocation matrix $U^{(2)}$ can be selected, because of minimum variance considerations say, and so on, where as if the complete distribution of the optimal objective function is estimated on the basis of large sample data with a fair degree of relaibility, then the optimum allocation matrix may have been U^*, where the optimum is defined by the utility functional (or risk preference functional), In case of optimum allocation matrix U^*, we do not have truncation at any stage, since we have complete specification of the probability distribution of the objective and also risk preferenee functional. In this aspect of evolving a sequential method of dealing with now observations, the active approach may be useful in suggestive criteria for changing from one optimum decision rule to anotherx as probability distribution of the objective function is specified less and less incompletely. In case of mixed and compound statistical distributions[4], the specification of which is rather complicated on the computational side, this may be difficult, yet very important.

1.4 FALLACY OF AVERAGES

An arbitrary non-linear function $f(x_1, x_2, - x_n)$ of random variables $x_1, x_2, -, x_n$ it is usually erroneous to

$$F\,[\,f(x_1, x_2, -, x_n)] = f\,(E[x_1] -, E(x_n)$$

Although mathematical analysis of a particular non-linear function may establish that its expected value is well approximated by the same function of the expected values.

A simplified example of an actual decision problem dealing with uncertain elements. It demonstrates how we can be dangerously misled by using average values in a model appropriate for a deterministic situation.

A company, which manufactures farm machinery, is planning to construct a new plant to build its latest equipment, a combine for harvesting, threshing, and cleaning grain.

Five major tasks must be completed in order to put the plant into full operation.

A. Erect plant building

B. Complete the final design of combine model.

C. Expand nucleus labour force to full-scale production size.

D. Install manufacturing equipment

E. Debug protype models.

Let tA, tB, ——, tE be the number of periods required for Tasks A, B, –, E. A period consists of three months. Assume that these tasks must be performed in the sequence indicated by Fig, 1.4. For example, both Tasks A and B can be started immediately. Both Tasks C and D can be started as soon as Task A is completed Task E can start when both tasks B and D are finished. The plant is in full scale production as soon as both Tasks C and E are completed.

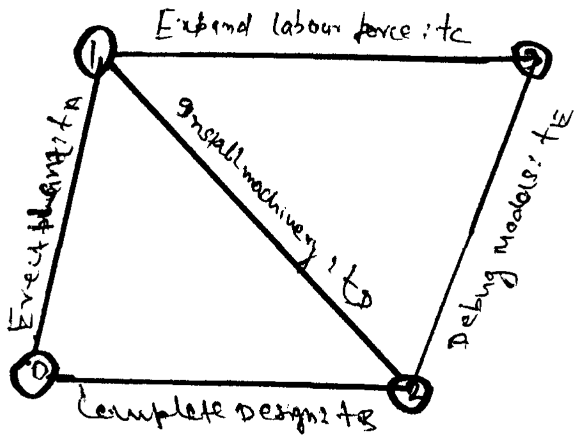

Fig : 1.4

It the values of tA, tB, ——tE were known with perfectly certainty, then the total span of the time required to put the plant into operation could be determined by finding the length of a longest route from Node 0 to Node 3 in the network of Fig. 1.4 several of these values are uncertain, however. Relying in part on past experience in constructing plants, company's Director has estimated the probability of each possible value, as such in Fig. 1.5. Thus it is equally likely that tB be either 2, 3 or 4 periods (that is, b, 9 or 12 months), similar statements hold for tC and tE. The times for tA and tD are known exactly. The director believes there is complete independence among the random events.

When each random event takes its smallest value, the total time span is 4 periods, and when each takes its largest value the span is 7 periods. Thus the range of completion times is between 12 and 21 months. If each random event takes its average value ($t_B = 3$, $t_C = 3$, $t_E = 2$)., the time span is 5 periods (15 months).

The director feel that there is significant compe titive advantage to having the plant in operation as soon as possible. His estimates of the incremental profit impact for different completion time of less than 7 periods are shown in Fig. 1.6. Being concerned about the drop in profit that occurs u the completion time stretches out to 6 or 7 periods, he consider the probability of hiring an experienced manager, to act as special assistant to supervise the construction project. He believes that the extra attention such a person could provide reduce the total time span by 1 period in any event. This is, even if the span would have been 4 periods, manager would cut the time to 3 periods and likewise for any other possible total span between 5 and 7 periods. The total cost of hiring this manager is Rs. 20,000. If the director knew for certain that the total span would be 4 or 5 periods, then he would decide against adding manager, Since the gain of Rs.10,000 would not offset the additional cost of Rs. 20,000. The opposite would be true director new for certain that the span would be 6 or 7 periods. Given the factual information in Fig. 1.4 through 1.6

Task	Possible Number of periods	Director's Assessed Probability
A	2	Perfect certainty
B	2,3,4	1/3 each
C	2,3,4	1/3 each
D	1	Perfect certainty
F	1,2, 3	1/3 each

Fig. 1.5

Total Time span(period)	Incremental Profit (Rs. 1000)
3	120
4	110
5	100
6	50
7	0

Fig. 1.6

<u>Analysis</u> : If analysis on the total span of 5 periods, calculated by using the averages time for each task, then we could not hire manager. The reason is that a reduction from 5 or 4 periods implies the profit gain would be Rs. 10,000 (= Rs. 110,000 - Rs 100,000) and this is less than the additional cost Rs. 20,000. Thus the analysis is quite faulty.

Such an approach makes two mistakes : first it assumes that the total time span calculated by looking at individual averages is a useful approximation to the expected total time span second it over looks the fact that the relevant criterion is expected incremental profit, and not incremental profit for the expected time span.

By evaluating all the different possible events and their probabilities of occurance, it can be shown that

$$P[T = 4] = 2/27,\ P[T = 5] = 8/27$$

$$P[T = 6] = 14/27,\ P[T = 7] = 3/27 \qquad \ldots.(1.4.1)$$

Where total T denotes the total time span.

There,

$$E\begin{bmatrix}\text{incremental profit}\\ \text{without new manager}\end{bmatrix} = (110)\ 2/27 + (100)\ 8/27 + (50)\ 14/27 + (0)\ 3/27 = 64. \qquad \ldots.(1.4.1)$$

and

$$E\begin{bmatrix}\text{incremental profit}\\ \text{with new manager}\end{bmatrix} = (120)2/27+(110)3/27 + (100)\ 14/27 +(50)3/27 = 99$$

To keep the arithmetic uncluttered, we have roudded the computations to the nearest thousand Rs. Thus the gain in expected incremental profit from the new manager is Rs. 35,000 (= Rs. 99,000 - Rs 64,000) which exceeds the additional cost because it was considerably over optimistic in assessing the economic outcome when the new assistant was not hired.

<u>Uncertainty and Information</u> : The economic impact of uncertainty is to evalute the gain in expected profit that would occur if company's direct were able to obtain a perfect prediction of the uncertain elements. In other words impact of the uncertainty can be measured as the maximum amount director would be willing to pay if, after the payment, he were able to learn the exact values of the random elements and consequently decide without error whether to hire the manager.

The decision to hire the manager is better only when T = 6 and T = 7, and the net profit will be Rs. 30,000 (= Rs. 100,000 - Rs. 70,000) and Rs. 30,000(= Rs. 50,000 - Rs. 20,000), respectively. Then the expected net profit when the values of the random elements are known prior to the decision is

$$E\begin{bmatrix}\text{not profit with}\\ \text{perfect information}\end{bmatrix} = (110)\ 2/27 + (100)\ 8/27 + (80)\ 14/27 + (30)\ 3/27$$

$$= 83 \qquad \ldots\ldots(1.4.4)$$

Given the expectation in (1.4.3), the expected net profit with uncertainty is Rs. 79,000 (= Rs. 99,000 - Rs. 20,000). Therefore the gain from perfect information is Rs. 4000 (= Rs. 83,000 - Rs. 79,000), taking account of the added cost of the new manager. This figure may be interpreted as measuring the impact or loss from uncertainty, for director would not pay over Rs. 4000 to have perfect foresight.

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Chapter - II

TWO STAGE STOCH PROGRAMMING PROBLEM

CHAPTER - 2

TWO STAGE STOCHASTIC PROGRAMMING MODEL

In this section we begin with the consideration how for probabilistic uncertainty in the coefficients of a linear programming problem.

Our task in the next several sections is to device ways of formulating so called stochastic programming model that yields ordinary linear programming problem model as a result. We can not hope to succeed in this task unless we add some specific postulates about the underlying structure of the decision process. In particular. We have to formalize the evaluation of which decision have to be made What information is known about previous decision and the random variables.' However, by examining a scaled-down version with only two stages to the decision process consequently that is the problem we consider this section.

In particular, we can learn two things from investigating the two stage model. First, we will see how to formulate a simple stochastic programming model to yeild an equivalent ordinary linear programming problem. Second, thing is that such a formulation magnifies the size of the problem.

An Easy Case : Before presenting an example of a two-stage model, we acknowledge a fundamental result for what might be formed a single one stage problem. To ease the exposition, suppose a deterministic version of the model is written in the canonical form.

$$\text{Maximize} \quad \sum_{j=1}^{n} C_i x_j \qquad \ldots.(2.1.1)$$

Subject to

$$\sum_{j=1}^{n} a_{ij}\, x_j = b_i \text{ fer } i = 1,2, - m \qquad \ldots (2.1.2)$$

$$x_j \geq 0 \qquad \text{fer } j = 1,2, - n \quad \ldots (2.1.3)$$

Now assume that the coefficient in the objective function really are random and that all the levels of variables here to determined prior to leaving the actual values for the variables prior to leaving for the random C_j, Such a situation might ~~be~~ arise in a planning model. Where ~~d~~ future market prices and labour costs are not known exactly at the time the plan is being developed. Since all the structural coefficients a_{ij} and the right hand. Side coefficient b_i are known with certainty, no difficulty arises in selecting feasible levels for the x_j.

LINEAR CERTAINTY EQUIVALENCE THEOREM

Assume that all the a_{ij} and b_i in (2.1.2) are known exactly, but C_j in (2.1.1) are random variables independent of all x_j. If the levels of x_j, for $j = 1, - n$ must be set prior to knowing the exact values of C_j then a solution to.

$$\text{Maximize } E\left[\sum_{j=1}^{n} C_j x_j\right] \qquad \ldots\ldots. (2.1.4)$$

Subject to (2.1.2) and (2.1.3) is given by levels for x_j that

$$\text{Maximize } \sum_{j=1}^{n} E[C_j]\, x_j \qquad \ldots\ldots(2.1.5)$$

Thus if the only random variables are the objective function coefficient, and these are independent of the specific activity levels, the an optimal solution can be found from an equivalent deterministic linear programme, where the corresponding expected values are used in the objective function. As we will see next, a linear problem with uncertainty is solved so simply when there are other random elements, or when there are same x_j that are set after learning the exact values for several of the random elements.

2.2 Two Stage Programming Technique

Before defining about the two-stage programming technique, first we go through stochastic linear programming problem which can be stated as follows.

$$\text{Minimize } f(X) = CTX = \sum_{j=1}^{n} C_j\, x_j \qquad \ldots.(2.2.1)$$

Subject to

$$A_i^T X = \sum_{j=1}^{n} a_{ij}\, x_j \geq b_i \quad i = 1, 2 - , m \qquad (2.2.2)$$

and

$$x_j \geq 0 \quad j = 1,2, \ldots n \qquad \ldots.(2.2.3)$$

where c_j, a_{ij} and b_i are random variables (the decision variables x_j are assumed to be deterministic for simplicity) with known probability distributions.

Two-stage programming technique is one which converts a stochastic LP problem into an equivalent deterministic problem. This is accomplished at the expanse of increasing the size problem. For simplicity, we assume that only the elements b_i are probabilistic.

This means that the variable b_i is not precisely known, but its probability distribution function, with a finite mean b_i, is known to us. In this case, it is possible to find a vector X in such a way that $A_i^T X$ will be greater than or equal to b_i $(i = 1,2,\ldots m)$ for whatever value b_i takes. In fact the difference between $A_i^T X$ and b_i will itself be a random variable, whose probability distribution function depends on the value of X chosen.

One can now think of associating a penalty of violation we might get for the constraints. In this case, we can think of minimizing the sum of $C^T x$ and the expected value of the penalty. There are several choices for the penalty, one choice is to assume a constant penalty cost of p_i for violating the ith constraint by one unit. Thus the total penalty is given by the expected (mean) value of the sum of individual penalties.

$\sum_{i=1}^{m} E(p_i y_i)$, where E is the expectation and y_i is defined as,

$$y_i = b_i - A_i^T X, \; y_i \geq \; i = 1,2, - , m. \qquad \ldots.(2.2.4)$$

Hence we can add the mean total penalty cost to the original objective function and write the new optimization problem as :

$$\text{Minimize } C_X^T + E\,(p^T Y) \qquad \ldots\ldots (2.2.5)$$

Subject to

$$A\,x + By = b \qquad \ldots..(2.2.6)$$

and $\quad X \geq 0, \; y \geq 0 \qquad \ldots..(2.2.7)$

where

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

and B = I Identify matrix oforder m notice that the penalty term in equ. (2.2.5) will be deterministic quantity in terms of the expected values of y_i, y_i. For example, if b_i follows uniform distribution in the range [$b_i - m_i$, $b_i + m_i$], and y denotes $b_i - A_i^T X$, then the mean penalty cost can be shown to be equal to.

$$E\;(p_i y_i) = p_{i1} + p_{i2} + p_{i3} \qquad \ldots.(2.2.8)$$

where $p_{i1} = 0$, if $y_i \geq m \qquad \ldots.(2.2.9)$

$$p_{i2} = \int_{S=0}^{(m_i - y_i)} \frac{p_i}{2m_i} \, sds, \text{ if } - m < y_i < m_i \qquad \ldots\ldots(2.2.10)$$

$$p_{i3} = \int_{S=(-m_i - y_i)}^{(m_i - y_i)} \frac{p_i}{2m_i} \, Sds, \text{ if } y_i \leq - m_i \qquad \ldots\ldots(2.2.11)$$

Thus we obtain

$$E(p_i y_i) = \frac{p_i}{4m_i} (m_i - y_i)^2 - p_i y_i \qquad \ldots\ldots(2.2.12)$$

which can be a quadratic function in terms of the deterministic variable y_i.

To convert the problem stated in equ. (2.2.4) to (2.2.7) to a fully deterministic one, the probabilistic constraints, Equ. (2.2.6) have to be written either in a deterministic form like $y_i = b_i - A_i^T X$, or interpreted as a two-stage problem as follows :

First Stage : First estimate or guess the vector X by solving the problem stated in Eqn. (2.2.1) to (2.2.3).

SECOND STAGE : Then observe the value of b and hence its discrepancy from the previous guess vector, and the vector $y = y\ (b, x)$ by solving the second stage problem :

Find y which minimizes $P^T y$

Subject to

$$y_i = b_i - A_i^T X, \; i = 1,2, - m$$

$$y_i \geq 0, \; i = 1,2, - , m \qquad \text{....(2.2.13)}$$

Where b_i and X are known now.

Thus the two-stage formulation can be interpreted to mean that a non-negative vector must be found (here and now) before the actual values of b_i $(i = 1,2, - m)$ are known, and that when they are known, a recourse y must be found by solving the second stage problem of Equns. (15). Hence, a general two stage problem can be stated as follows :

Minimize

$$C^T CX + E\,[\min_{y}(P^T y)]$$

subject to $\qquad$ ....(2.2.14)

$$\underset{m \times n_1}{A}\;\underset{n_1 \times 1}{X} + \underset{m \times n_2}{B}\;\underset{n_2 \times 1}{y} \geq \underset{m \times 1}{b}$$

$$X \geq 0, \; y \geq 0$$

2.3 <u>The two stage stochastic programming problem with recourse</u> :

Let

$$\psi = \min Cx + \sum_{r=1}^{n} Pr\,(h^r\, y^r) \qquad \text{....(2.3.1)}$$

Subject to

$$A^o X \leq b^o \qquad \text{....(2.3.2)}$$

$$A^r X + O^r y \leq b^r \text{ where } r = 1,2 - N \qquad \text{....(2.3.3)}$$

$$X \geq 0, \; y^r \geq 0 \quad \text{where } r = 1,2,..N \qquad \text{....(2.3.4)}$$

Where p^r are probabilities, such that

$$\sum_{r=1}^{N} Pr = 1 \quad \text{and } Pr \geq 0 \text{ for } r = 1, -- N$$

The first stage decision variables in this model are the variables. When they have been chosen, there are N possible result of data h^r, Ga^r and b^r each outcome occuring with probability P_r, y^r are used the achieved feasibility on the constraints (2.3.3) in the second stage of decision making after the random outcomes have occured. The objective is to minimize the expected cost of the first and second stage decision.

The stochastic programming problem can be solved with recourse by using Bender's algorithm. In partitioned promas,

$$\psi = \min CX + \sum_{r=1}^{N} \psi\, r(X)$$

Subject to $A^o X \leq b^o$

and $X \geq 0$

where, for any ψ we have the following sub problem.

$$\psi\, r\, (X) = \min Pr\, (\psi^{r} y^r)$$

Subject to $Q^r y^r \leq b^r - A^r X$ ....(2.3.6)

$$y^r \geq 0$$

The equivalent of (2.3.6) will be follows :

$$\psi^r (X) = \max V^r(b^r - A^r X)$$

Subject to $V^r Q^r \leq P_r h^r$

$$V^r \leq 0 \qquad \text{....(2.3.7)}$$

The basis for the construction of a master problem of the linear convex function ψ^r is represented by (2.3.7)

Example :

A factory has 100 items on hand which may be shipped to an outlet at the cost of \$ 1 a piece to meet an uncertain demand di. In the event that the demand should exceed the supply. It is necessary to meet the unsatisfied demand by purchases on local market at \$ 2 a piece. The equations that the system must satisfy are.

$$100 = X_{11} + X_{12}$$

$$d_2 = X_{11} + X_{21} - X_{22} \quad (X_{ij} \geq 0) \quad - \text{(a)}$$

$$C = X_{11} + X_{21}$$

where X_{11} = number shipped from factory

X_{12} = number of stored at factory.

X_{21} = purchased on open market

X_{22} = excess of supply over demand

d_2 = unknown demand uniformly distributed between 70 and 80.

C = total cost.

It is clear that whatever be amount shipped and whatever be the demand d_2, it is possible to chose X_2, and X_{22} consistent with second equation. The used stocks X_{12} + X_{22} are assumed to have no value or are written off at some reduced value.

<u>Solution</u> : Let us consider the two stage case given by (as it is clear that if supply exceeds demand ($X_{11} > d_2$) that $X_{21} = 0$ gives minimum cost. Hence,

$$\min_{X_{21}} \phi = \begin{bmatrix} X_{11} & \text{if } X_{11} > d_2 \\ & \\ X_{11} + 2\,(d_2 - X_{11}) & \text{if } X_{11} \leq d_2 \end{bmatrix}$$

When $70 \leq X_{11} \leq 80$ we have

$$\text{Exp}[\min \phi] = 1/10 \int_{70}^{X_{11}} X_{11} dd_2 + 1/10 \int_{X_{11}}^{80} X_{11} + 2(d_2 - X_{11}) dd_2$$

$$= \left| \frac{1}{10} \, dX_{11} \right|_{70}^{X_{11}} + \left| \frac{1}{10} \, \frac{d^2}{2} \right|_{X_{11}}^{80} - \left| \frac{1}{10} \, X_{11} \, d \right|_{X_{11}}^{80}$$

$$- 2\,(d_2 - X_{11})$$

$$= 77.5 + \frac{1}{10} (75 - X_{11})^2$$

when $X_{11} \leq 70$, we have

$$\underset{d_2}{\text{Exp}} \left[\underset{X_{21}}{\min} \emptyset \right] = \frac{1}{10} \int_{70}^{80} X_{11} + 2\,(d_2 - X_{11})$$

$$= - X_{11} + 150$$

Therefore,

$$\underset{d_2}{\text{Exp}} \left[\underset{X_{21}}{\min} \emptyset \right] = \begin{bmatrix} -X_{11} + 150 & \text{if } X_{11} \leq 70 \\ 77.5+1/10(75-X_{11})^2 & \text{if } 70 \leq X_{11} \leq 80 \\ X_{11} & \text{if } 80 \leq X_{11} \end{bmatrix}$$

This function is closely conven and attains its minimum at 77.5, which is expected cost, at $X_{11} = 75$. Since $X_{11} = 75$ is in range of possible values of X_{11} as defermined by $100 = X_{11} + X_{12}$. This is clearly the optimal shipment on this case it pays to ship $X_{11} = d_2 = 75$ the expected demand.

2.4 Application for two stage stochastic programming

Stochastic Programming Models For Scheduling Airlift Operations :

Scheduling airlift operations is most applicable in two stage stochastic linear programms. This model consists of two components - a monthly and a daily flight planning model. Both are formulated as stochastic two-stage linear program's, and can be solved by algorithms based upon standard limear calculations, consequently, the model can be applied to large scale systems. This should permit sheduling the system as a whole rather than the two divisions separately. With transfers of aircraft between then on an adhoc basis, as is currently the practice. Here we will discuss a monthly flight pâanning model only.

Description of the Airlift System

Air lift missions within the system are two basic types channel and special missions. Channel missions fly on a regular basis between fixed origins and destinations. Fixed airports in the Indian component aerial ports of embarkation (APOEs) and oterseas bases, and aerial ports of debarkation (APODs). They carry cargo and passengers. They are scheduled in advance for each calendar month, Although the schedules (number of flights and departure times) are changed during the month, often daily, based upon deviations from expected cargo generation many of these flights make intermediate stops for re-fueling and crew changes. Channel cargo is shipped into APOEs for storage until shipped.

Apecial missions are flights from points of original cargo generation, such as factories or warehouses, to points at or hear the ultimate consignee. Such flights ariee on an irregular basis, and are scheduled if sufficient advance notice is provided. The time and place of cargo pickup is usually specified. Often these flights arise after the begining of the month and require a temporary diverting of aircarft from channel missions. Since certain special missions are designated higher priority than channel traffic. The users of a special mission must, in effect, charter an entire aircraft, although the airlift operator decides what type of aircraft will be supplied.

MATHEMATICAL FORMULATION

Here we formulate the monthly and daily models as two-stage linear programs under uncertainty[17][57] with random right hand side in the second-stage. Several of the program's variables represent number of different kinds of flights and consequently must be integer valued. The general form for problems of this type is the following.

$$\min_{x \geq 0} (x + E \ [\min_{y \geq 0} q \ y | W_y = \ - T_x]$$

$$\text{subjecte} \quad \text{to } A_x = b \qquad \ldots.(2.4.1)$$

In this problem, c, q, and b are known vectors of dimension n_1, n_2, and m_1, respectively. A, T, and W are known matrices of dimension m_1 X n_1, m_2 x n_1, and m_2 x n_2, respectively. The vectors x and y represent the first and second-stage decision vectors. is a random vector with a known probability distribution and E(.) denote the expectation with respect to that distribution.

Monthly Model

In the monthly model the sequence of decision for flight route is the following :

(1) The number of flights of each aircraft in each route is assigned before monthly requirements. Each route are known with certainly.

(2) After monthly requirements are observed, some flights assigned to routes with lower than expected demand are switched to routes with higher than expected demand. Commercial airlift is also sometimes added to routes with excess demand.

This model has the following structure. The first-stage constraints state that for each aircraft type total number of flying hours allocated to all routes cannot exceed the total number of flying hours available of that type. The second stage constraints are of two types. The first specification that the no. of flying hours of a given air craft type diverted from a particular route to other routes cannot exceed those initially assigned to it. The second type are demand balance equations, which state that for each route the total carrying capacity (that originally assigned minus that diverted to other routes plus that diverted from other routes) minus unused carrying capacity plus unsatisfied demand is equal to total demand for that route.

The objective function consists of the cost of the final flying program (the initial plus the amended assignment) plus penalty costs of excess demand or supply. The cost of access demand is reflected in both the cost of additional commercial lift plusthe extra flying time consumed in switching aircraft from one route to another. Specifically, the program is as follows :

Final min Z, x_{ij}, x_{ijk}, y_j^+ y_j^- ≥ 0 such that

$$\sum_{i,j} C_{ij} x_{ij} + E \left\{ \min \left[\sum_{i,j,k \neq j} \left(C_{ijk} - C_{ij} \frac{a_{ijk}}{a_{ij}} \right) x_{ijk} + \sum_{i} C_j^+ y_j^+ + \sum_j C_j^- y_j^- \right] \right\} = Z, \quad \text{—— (2.4.2)}$$

First stage

$$\sum_j a_{ij} \; x_{ij} \leq F_i, \qquad \text{all}_i \qquad \text{—— (2.4.3)}$$

Second stage

$$x_{ij} - \sum_{K \neq j} \frac{a_{ijk}}{a_{ij}} x_{ijk} \geq 0 \; , \qquad \text{all } i,j \text{ —— (2.4.4)}$$

$$\sum b_{ij} x_{ij} - \sum_{i,k \neq j} \left[b_{ij} \left(\frac{a_{ijk}}{a_{ij}} \right) \right] x_{ijk} +$$

$$\sum_{i,K \neq j} b_{ij} \; x_{ijk} - y_j^+ - y_j^- = d_j \; , \text{ all } j \qquad \text{—— (2.4.5)}$$

where x_{ij} = number of flights of air craft type i initially assigned to and flown on rout j;

x_{ijk} = number of flights of aircraft type i assigned to route K using hours made available by canceling route, flights,

y_j^+ = demand on route j which is satisfied by commercial lift, if permitted, or unsatisfied demand if commercial lift not permitted,

y_j^- = unused capacity on route j;
and

a_{ij} = number of hours required by aircraft type i for a flight initially assigned to and flown on route j,

a_{ijk} = number of hours required by aircraft type i for a flight on route K that uses hours made available by canceling route j flights ($a_{ijk} \geq a_{ik}$);

b_{ij} = the carrying capacity (tons or any other appropriate measure) of a flight of an aircraft of type i on route j,

F_i = maximum number of flying hours for aircraft of type i available during the month;

d_j = total demand (tons or any other measure) for route j (a random variable),

C_{ij} = Cost of flight of aircraft type i initially assigned to and flown on route j ($C_{ij} \geq 0$),

C_{ijk} = Cost per flight of aircraft type i assigned to route K from hours made available by canceling route j flight ($C_{ijk} \geq C_{ik}$)

C_j^+ = Cost per ton of commercial augmentation on route j, If commercial augmentation is not available to carry excess demand, t is may instead represent a shortage cost,

C_j^- = Cost of a unit of unused carrying capacity on route j.

Since a flight assigned to and flown on ourte K from hours diverted from route j takes a_{ijk} hours, such a flight on route K results in the cancellation of (a_{ijk}/a_{ij}) flight on route j. Thus, of the flights initially assigned to route j, those that are actually flown number

$$x_{ij} - \sum_{K \neq j} \frac{a_{ijk}}{a_{ij}} \quad x_{ijK}$$

This is reflected in expression (2.4.2), (2.4.3) and (2.4.4) The first-stage constraints correspond to

$Ax = b$ and second-stage constraint to

$Wy = - T_x$ in expression (2.4.1).

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Chapter - III

CHANCE CONSTRAINED PROGRAMMING

CHAPTER - 3

3.1 CHANCE CONSTRAINED PROGRAMMING PROBLEM

Chance constrained programming problem is one which can be used to sølve problem involving chance constraints, i.e. constraints having finite probability of being violated. This chance constrained programming permits the constraints to be violated, by a specified (small) probability. This problem was initially studied by A.charnes and W.W. Cooper[10], when the first works investigating stochastic programming chance consstrained problem appeared in 1960 and the works by Charnes-Cooper and Symond[7] were published. Moreover different problems for the qualitative analysis of chance constrained problems were contributed by Miller and Wegner [34]. Sengupta [46][47] and others.

In stochastic programming problem some constraints may be deterministic and the remaining may involve random elements. Whereas in chance counstrained programming problem the latter set of constraints is not always required to hold, but these must hold simultaneously or in individually with given probabilities. In otherwords, we are given a set of probability measures indicating the extent of violation of the random constraints. The general chance constrained linear program is of the form.

$$\text{Minimize } f(X) = \sum_{i=1}^{n} C_j X_j \qquad \ldots(1.2.1)$$

subject to $P[\sum_{j=1}^{n} a_{ij} X_j \leq b_i] \geq p_i,$

$i = 1,2 \ldots m$ ...(1.2.2)

and $X_j \geq 0,\ j = 1,2, \ldots n$ ...(1.2.3)

Where C_j, a_{ij} and b_i are random variables and p_i are specified probabilities. Where $0 \leq p_i \leq 1$. Symond J.[48] formulated conditions of deterministic equivalent to chance constrained stochastic problems, and weshels J. [56] investigated the condition of convexity for a deterministic equivalent. The subject of chance constrained programming was further extended and applied by charnes and Cooper [7a], [7b],[7c], Charnes, Cooper and Thomson [9], Tataoka [24], Kirby [25], Naslund[36]. Naslund and Whinston [37], Sinha, van De Panne and Popp [40], and Hiller [26], Poliys [39] gave a new approach to the solution of chance constraints problem by applying iteration methods.

Application of chance constraints stochastic programming problems into various field such as transportation problem study of Agricultural production. Air trafic control, Functioning and production out-put for an industry. etc. were considered by Lavirnenka[27].Friendland[18], Judin [22] and others, survey works including chance constrained stochastic programming problems were taken by Zetmer[58] and Judin[23] etc.

3.2 CHANCE CONSTRAINED PROGRAMMING TECHNIQUE

As the name indicates, chance constrained programming technique is one which can be used to solve problems involving chance constraints, that is, constraints having finite probability of being violated. This chance constrained programming permits the constraints to be violated by a specified (small) probability where as two-stage programming does not permit any constrained programming technique was originally developed by chamned and Cooper.

In chance constrained programming the stochastic linear programming problem is stated as follows :

Minimize

$$f(X) = \sum_{j=1}^{n} C_j X_j \qquad \ldots(3.2.1)$$

Subject to

$$P\left[\sum_{j=1}^{n} a_{ij} x_j \leq b_i \right] \geq P_i, \; i=1,2,\ldots m \qquad \ldots(3.2.2)$$

$$x_j \geq 0, \; j = 1,2, \text{—} n \qquad \ldots(3.2.3)$$

where C_j, a_{ij} and b_i are random variables and p_i are specified probabilities.

Notice that Eqs (3.2.3) indicate that the its constrain

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

has to be satisfied with probability of at least p_i where $0 \leq p_i \leq 1$. For simplicity we are assuming that the decision variable x_j are deterministic. We shall first consider special cases where only C_j, a_{ij} or b_i are random variables before considering the general case in which c_j, a_{ij} and b_i are all random vatiables. We shall further assume that all the random variables are normally distributed with known mean and standard deviations.

(i) <u>When only a_{ij} are random variables</u> :

Let a_{ij} and var $(a_{ij}) = \sigma^2 a_{ij}$ be the mean and the variance of the normally distributed random variable a_{ij} Assume that the multivariate distribution of a_{ij}, $i=1,2..m$, $j = 1...n$ is also known alongwith the covariance Cor $(a_{ij}$, akl) between the random variables a_{ij} and akl.

Define quantities di as

$$d_i = \sum_{j=1}^{n} a_{ij} x_j, \; i = 1,2..., m \qquad ...(3.2.4)$$

Since a_{i1}, a_{i2}, $...a_{in}$ are normally distributed, and x_1, $x_2, ...x_n$ are constants (not yet known) d_i will also be normally distributed with a mean value of

$$d_i = \sum_{j=1}^{n} a_{ij} x_j, \; i=1,2,...m \qquad ...(3.2.5)$$

and a variance of

$$\text{var } (d_i) = \sigma^2_{d_i} = X^T V_i X \qquad ...(3.2.6)$$

where V_i is the ith covariance matrix defined as

$$V_i = \begin{bmatrix} Var(a_{i1}) & COV\ (a_{i1}, a_{i2}) & ....COV(a_{i1}, a_{in}) \\ Cov(a_{i2}, a_{i1}) & Va(a_{i2}) & ....Cov(a_{i2}, a_{in}) \\ Cov(a_{in}, a_{i1}) & Cov(a_{in}, a_{i2}) & ....Var(a_{in}) \end{bmatrix}$$

The constraint of Eqn (3.2.2) can be expressed as

$$P\,[d_i \leq b_i] \geq p_i$$

ie, $$P\left[\frac{d_i - \bar{d}_i}{\sqrt{var(d_i)}} \leq \frac{b_i - \bar{d}_i}{\sqrt{var(d_i)}}\right] \geq p_i, i=1,2,..m \qquad ...(3.2.7)$$

where $(d_i - \bar{d}_i)/\sqrt{var(d_i)}$ can be seen to be a standard normal variat with mean of zero and a variance one. Thus the probability of realizing d_i smaller than or equal to b_i can be written as

$$P\,[\,d_i \leq b_i] = \phi\left(\frac{b_i - \bar{d}_i}{\sqrt{var(di)}}\right) \qquad ...(3.2.8)$$

where $\phi(x)$ represents the cumulative distribution function of the standard normal distribution evaluated at x. If ei denotes the value of the standard normal variable at which

$$\phi\,(ei) = p_i \qquad ...(3.2.9)$$

then the constraints in Eq.(3.2.7) can be stated as

$$\phi\left(\frac{b_i - \bar{d}_i}{\sqrt{var(di)}}\right) \geq \phi(ei),\ i=1,2,...m \qquad (3.2.10)$$

or

$$\bar{d}_i + ei\sqrt{var(di)} - b_i \leq 0, i=1,2, ...m \qquad ...(3.2.11)$$

By substituting Eqs (3.2.5) and (3.2.6) in Eq.(3.2.11), we obtain

$$\sum_{j=1}^{n} \bar{a}_{ij}\, x_j + Ci\,\sqrt{X^T V_i\, x} - \bar{b}_i \leq 0, \qquad (3.2.12)$$

$$i = 1,2...m$$

There are deterministic non-linear constraints equivalent to the original stochastic linear constraints.

$$V = \begin{bmatrix} Var(c_1) & Cov(C_1,c_2) & \text{---} & COV(C_1,C_n) \\ COV(C_2,C_1) & var(C_2) & \text{---} & COV(C_2,C_n) \\ \vdots & \vdots & & \vdots \\ COV(C_n,C_1) & COV(C_n,C_2) & \text{---} & Var(C_n) \end{bmatrix} \quad \text{---}(3.2.23)$$

with $Var(C_j)$ and $COV(C_i,C_j)$ denoting the variance of C_j and Covariance between C_i and C_j respectively.

A new deterministic objective function for minimization can be formulated as

$$F(X) = K_1 f + K_2 \sqrt{Var(f)} \qquad \ldots(3.2.24)$$

Where K_1 and K_2 are nonnegative constant whose values indicates the relative importance of f and standard deviation of f for minimization. Thus $K_2 = 0$ indicates that the expected value of f is to be minimized without caring for the standard deviation of f. On the other hand, if $K_1 = 0$ it indicates that we are interested in minimizing the variability of f about its mean value without bothering about what happens to the mean value of f. Similarly, if $K_1 = K_2 = 1$, it indicates that we are given equal importance to the minimization of the mean as well as the standard deviation of f. Notice that the new objective function stated in Eq.(3.2.24) is a non-linear function in X in view of the expression for the variance of f.

Thus the solu stochastic linear programming problem stated in Eqs(2.2.1) to (2.2.3) can be obtained by solving the equivalent deterministic nonlinear programming problem.

Minimize $F(X) = K_1 \sum_{j=1}^{n} C_j x_j + K_2 \sqrt{X^T V X}$

Sübject to

$$\sum_{j=1}^{n} a_{ij} x_j - b_i \leq 0 \quad i=1,2,\ldots m \qquad \text{——(3.2.25)}$$

and

$$x_j \geq 0, \; j = 1,2, \ldots n$$

If all the random variables c_j are independent, the objective function reduces to

$$F(X) = K_1 \sum_{j=1}^{n} C_j \, x_j + K_2 \sqrt{\sum_{j=1}^{n} \mathrm{var}(C_j) x_j^2} \qquad \text{——(3.2.26)}$$

iv) <u>When C_j, a_{ij} and b_i are random variables</u> :

As the random variables C_j, $J = 1,2,\ldots n$ appear only in the objective function, we x can take the new objective function $F(X)$ same as the one given in Eq.(3.2.24). The constraints of Eqn.(3.2.2) can be expressed as

$$P[h_i \leq 0] \geq P_i \quad i = 1,2 \ldots m, \qquad \text{——(3.2.27)}$$

where h_i is a new random variable defined as

$$h_i = \sum_{j=1}^{n} a_{ij} x_j - b_i = \sum_{K=1}^{n+1} q_{ik} y_k \qquad \text{——(3.2.28)}$$

where

$$q_{ik} = a_{ik}, \; K = 1,2,\ldots n$$

$$q_{i,n+1} = b_i .$$

$$y_K = x_k, \quad K = 1, 2, \text{——} n$$

and

$$y_{n+1} = -1$$

Notice that the constant Y_{n+1} is introduced for convenience, Since h_i is given by a linear combination of the namely distributed random variables q_{ik} it will also follow normal distribution. The mean and the variance of hi are given by

$$\bar{h}_i = \sum_{K=1}^{n+1} q_{ik} Y_k = \sum_{j=1}^{n} a_{ij}\, x_j - b_i \qquad \text{——}(3.2.29)$$

and $\mathrm{Var}(h_i) = y^T V_i y$ ——(3.2.30)

$$Y = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{Bmatrix}$$

and

$$V_i = \begin{bmatrix} V(q_{i1}) & COV(q_{i1}, q_{i2}) & \text{———} & (COV(q_{i1}, q_{i,n+1}) \\ COV(q_{i2}, q_{i1}) & var(q_{i2}) & \text{———} & COV(q_{i2}, q_{i,n+1}) \\ COV(q_{i,n+1}, q_{i1}) & COV(q_{i,n+1}, q_{i2}) & \text{-.} & Var(q_{i,n+1}) \end{bmatrix}$$

This can be written more explicity as

$$\mathrm{Var}\,(h_i) = \sum_{K=1}^{n+1} \,[\, y_k^2\, \mathrm{Var}(q_{ik}) + 2 \sum_{l=K+1}^{n+1} y_k y_l COV(q_{ik}, q_{il})$$

$$= \sum_{K=1}^{n} [y_K^2\, var(q_{ik}) + 2 \sum_{L=K+1}^{n} y_k y_l\, COV\,(q_{ik}, q_{il})]$$

$$+ y_{n+1}^2\, \mathrm{Var}\,(q_{i,n+1}) + 2y_{n+1} COV(q_{i,n+1}, q_{iN+1})$$

$$+ \sum_{K=1}^{n} [2\, y_k y_{n+1}\, \mathrm{COV}\,(q_{ik}, q_{i,n+1})]$$

$$= \sum_{K=1}^{n} [\, x_K^2 \;\; \mathrm{Var}(a_{ik}) + 2 \sum_{L=K+1}^{n} x_k \;\mathrm{Cov}\,(a_{ik}, a_{il})]$$

$$+ \;\mathrm{Var}(b_i] - 2 \sum_{K=1}^{n} x_k \;\mathrm{Cov}\;(a_{ik}, b_i) \qquad \text{—(3.2.31)}$$

Thus the constraints in Eqs (3.2.27) can be restated as

$$P\left[\frac{h_i - h_i}{\sqrt{\mathrm{Var}(h_i)}} \leq \frac{-h_i}{\sqrt{\mathrm{var}(h_i)}}\right] \geq P_i$$

$$i=1,2\text{—}m \qquad \text{—(3.2.32)}$$

where $[(h_i - h_i)\;(\sqrt{\mathrm{Var}(h_i)}]$ represents a standard normal variable with a mean value of zero and variance of one.

Thus if ei dendes the value of the stanaard normal variable at which

$$\phi\;(e_i)=P_i \qquad \text{—3.2.33)}$$

the constraints of Eq.(3.2.32) can be stated as

$$\phi\;(-\frac{h_i}{\sqrt{\mathrm{var}(h_i)}} \geq \phi\;(e_i),\; i = 1,2,\ldots m \quad \ldots(3.2.34)$$

These inequalities will be satisfied only it following deterministic non-linear inequalities are satisfied.

$$\frac{-h_i}{\sqrt{\mathrm{Var}(h_i)}} \geq e_i \quad l=1,2,\ \ldots m$$

$$h_i + e_i\;\sqrt{\mathrm{Var}\;(h_i)} \leq 0,\; i = 1,2,\;\ldots m \qquad \ldots(3.2.35)$$

Thus the stochastic linear prog. problem of Eqn (2.2.1) to (2.2.3) can be stated as an equivalent deterministic nonlinear prog. problem as minimize

$$F(X) = K_1 \sum_{j=1}^{n} C_j x_j + K_2 \sqrt{X^T V X}, K_1 \geq 0, \ K_2 \geq 0$$

Sùbject to

$$\bar{h}_i + e_i \sqrt{\mathrm{Var}(h_i)} \leq 0 \quad i=1,2, \ \ldots m$$

and $x_j \geq 0, \ j = 1,2,\ldots n$ ...(3.2.36)

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3.3 CHANCE CONSTRAINED GENERALIZED NETWORK

Here we consider the extensions of the current theory of generalized network problems (from a linear programming viewpoint) to cover situations in which the nonzero entries of the generalized incidence matrix may be random variables. The extension involves an interpretation of the constraints as chance constrained and thereby the extended problem becomes a chance-constrained programming problem we solve this problem for the optimal zero-order rule and, in doing so, we obtain a chance-constrained problem that is the dual of our original problem. This dual chance-constrained problem is obtained through the use of dual 'deterministic equivalent's. Thus, we extend to this class of models, the chance constrained duality theorem (3.3.2), which allowed random variables only in the stipulations vector.

The interpretation of this dual problem and show how it can be used to solve the given problem we also show that our results hold regardless of the distributions of the random variables involved in the problem. Finally, we indicate how similar techniques can be used to handle the case in which the optimal objective function is also stochastic in nature. After investigation of this problem was motivated by consideration of the problem of optimal design of wastewater treatment plants discussed in reference[53]. In this problem, the nonzero elements ε_j in the generalized incidence matrix represent 'process factors' associated with the jth process. In particular, ε_j is the factor by which the amount of flow out of process j differs from the flow into process j as a result of the flow undergoing the jth process. From the nature of the problem it is cleared that ε_j is a random variable, since the efficiency with which the process operates depends on such stochastic quantities as the density or composition of the flow, the temperature of the treatment chambers and ingredients etc.

Hence ε_j is not constant but rather fluctuates in some way over a range of possible values. Thus, it is not possible in general to specify a flow pattern, in advance, that will be optimal (or even physically feasible) for all possible values of the ε_j. However, we can find a flow pattern that is both feasible and optimal within certain preassigned probability limit. This type of interpretation leads to constraints that are conveniently expressed in the farm of chance constraints.

We will solve our chance-constrained problem for the optimal zero order rule. Zero order rules are used in reference [9] in a discussion of PERT-type scheduling problems in which the dual stipulations vector project completion times is assumed to be random.

Generalized Network Problems: Deterministic Case :

A(pure) Network is an oriented connected graph with the following additional features: associated with each Link (or arc) is not only a direction but a unit cost of flow and associated with each node (or vertex) is a quantity representing an influx or efflux. Flow is regarded is taking place along the links from nodes at which influx is present to nodes at which efflux is to occur: flow on any link incurs a per unit cost in an amount given by the cost associated with the link. Capacitated networks, meaning networks in which there is an upper bound to the flow on each link, are not considered here.

Such a network, having m nodes and n links, can be described by its incidence matrix A, an m x n matrix in which the jth column (corresponding to link j) contains -1 in row k, + 1 in row q_1 and zeros elsewhere when link j leads from node K to node q. A m-vector, b, contains in its ith position the influx (with a-sign) or efflux (with a + sign) associated with node i : the n-vector, C, contains in its jth position the unit cost associated with link j.

If it is desired to minimize total cost while satisfying the influent and effluent restrictions, the optimal flow pattern x is the sdution to the linear programming problems

$$
\begin{array}{lll}
\text{Minimize} & C^T x & \\
\text{Subject to} & A_x = b, & \quad \ldots (3.3.1) \\
 & x \geq 0, &
\end{array}
$$

Where $x = \begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix}$ x_j is flow on link j, and the ith constraint is a statement of the kirchoff conservation condition at the ith node. There exist many variations on this theme (e.g.some of equations in (3.3.1) may be replaced by inequalities) but all such variations can be converted into problems with structure of (3.3.1). A generalized network differs from the above in that the nonzero entries in A are not required to be ± 1, although it is still required that each column have exactly two nonzero entries that are of opposite sign. It is clear that, by appropriate scaling of the columns of A and the corresponding elements of C, an equivalent problem may be obtained in which the negative element in each column of A is equal to -1, The positive element in the jth column of A will be denoted k_j. The flow on link j may be regarded as incurring a cost of $C_j x_j$ and then being subjected to amplification (or attenuation) by the factor K_j . Thus the flow along link j is amplified by a factor K_j during the course of its transversal of link j. This is in contrast to the pure network case where all $K_j = 1$.

Thus dual to (3.3.1) is

Maximize $W^T b$,

Subject to $W^T A \leq C^T$ ...(3.3.2)

In order to understand the meaning of dual constraints we proceed as follows, First, note that the dual variable w_i is associated with node i and there is one dual constraint for each link (i,j) in the network. Second, recall (from linear programming theory) that a basic optimal solution to (3.3.2) will have at a least m constraints satisfied as equalities, any m such equations serve to specify a basic solution to the primal problem.

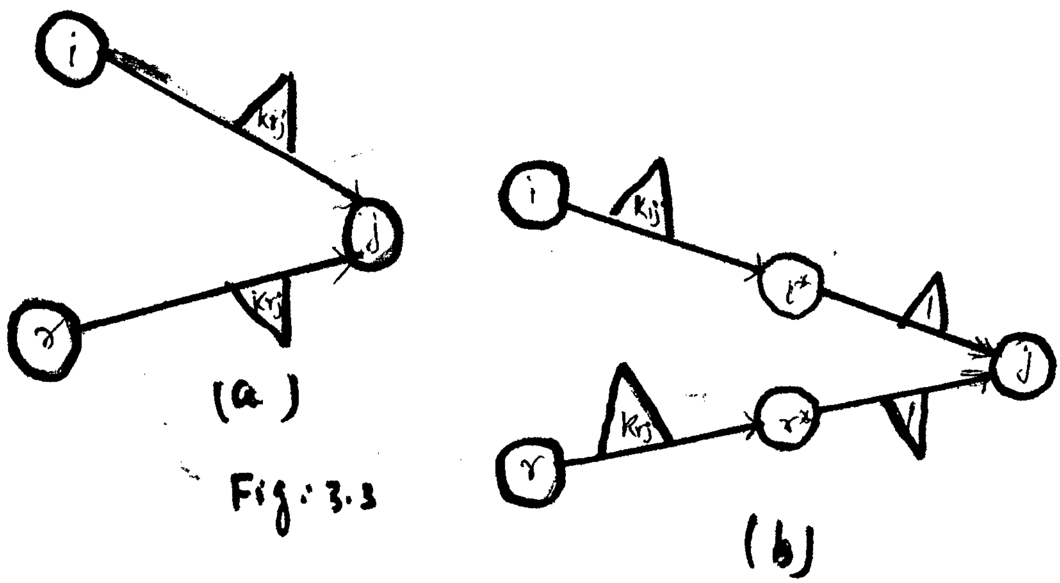

Fig: 3.3

Satisfaction of the remaining constraints of 3.3.2 is, in fact, a criterion for optimality of corresponding basic solution to (3.3.1) suppose now that we were to interpret wi as representing the per-unit net decrease in cost that would be obtained, supposing that it were feasible to increase the flow out of node i by modifying the flows on links of the network that are incident on node i. The we are ƒ thus virtual quantities in the same sense that virtual quanties appear in other fields such as mechanics, etc since, for any j, increasing the flow along link (i,j) by one unit involves increasing the flow into node i by one unit and increasing the flow out of node j by k_{ij} units, the per unit net decrease in cost that would be obtained by such a change is $k_{ij}\ W_j +(-w_i) = K_{ij}W_j - W_i$. Thus, under this interpretation of the dual variables, w_i, the dual constraints state that the (per unit) net decrease or 'virtual decrement' in cost that would be obtained by increasing the flow link(i,j) by one unit must be less than or equal oto the per unit-actual cost increase, C_{ij}, which would be incurred by such a change, i.e., $K_{ij}\ W_j - W_i \leq C_{ij}$ for each link (i,j). Hence a flow pattern is optimal (i.e. results in a set of W_i that satisfy the dual constraints) if and only if a change in the flow along any link would cost more than benefits that would be obtained from such a change. This then is the interpretation we will give to the dual variables and constraints.

Algebrically this means that, with no loss of generality, the matrix A in (3.3.1) can be assumed to have at most one positive entry per row which is not equal to 1. To see this refer to Fig.3.3 in which the nodes i^* and r^* are adjained to the original network, in order to convert the original network shown in 1 as to the one in (b), which has the required property.

3.4 APPLICATIONS OF CHANGE CONSTRAINED STOCHASTIC PROGRAMMING PROBLEM :

An example of the agricultural production planning model which represents the chance constrained stochastic linear programming problem is given here.

The following notations are defined :

m is the number of crops ($1 \leq r \leq m$)

n_r is the total number of sorts

t is the number of types of soil in which the experiment is planned ($1 \leq j \leq t$).

g_j is the given area of soil of type j,

p_r is the planned volume for the production of rth crop,

$1-d_r$ is the addmissible risk for non fulfilment of the planned production of rth crop.

r_{kj} is the crop capacity of the sort K of r crop on j soil.

C_{rkj} is the expenditure on the cultivation per unit of j soil for the saving of K sort of r crop.

1-r is the addmissible risk with which the actual expenditure may be more than that planned.

Y_{rkj} is the area occupied by K soil of r crop on j soil.

In the defined notations, the agricultural production planning problem is presented as follows :

$$\text{Minimize} \quad Y$$

$$\text{S.t} \quad P\left\{\sum_{r=1}^{m}\sum_{K=1}^{nr}\sum_{j=1}^{t} C_{rkj}\, Y_{rkj} \leq Y\right\} = r$$

$$P\left\{\sum_{K=1}^{nr}\sum_{j=1}^{t} r_{kj}\, Y_{rkj} \geq P_i\right\} \geq \alpha_r$$

$$(1 \leq r \leq m)$$

$$\sum_{r=1}^{m}\sum_{K=1}^{n_r} Y_{rkj} = g_j, \ (1 \leq j \leq t)$$

$$Y_{rkj} \geq 0, \ r = 1, \quad m, \ K = 1, \ldots n_r$$

$$j = 1, \quad t.$$

The applications of planning and management mathematical methods under uncertainty are used in the conditional extremum problems in which there are not only probabilistic but also statistical and rigid constraints. The deterministic equivallence of problems with random parameter with chance constrained models, represent, in general, nonlinear and some times even non-convex, even typical bivary programming problems. Hence in stochastic programming it is usually not important whether the problem is the result of linear as a non linear or bivary expremum problem.

Literature Survey of CCP Application

[References] program Description	Nature of chance constraineds
[1] Mimimizin cost of staffing a hospital	Probability that there will be a demand overload for hospital staff not more than $\alpha\%$ of the time.
[2] Profit maximization in a capital budgeting framwork	1. Probability that return on investment will be less than a special amount only $\alpha_1\%$ of the time. 2. Probability that liquidity measures will be less than a specified level no more than $\alpha_2\%$ of the time. 3. Probability that cash demand will note exceed cash reserves more than $\alpha_3\%$ of the time.
[6] optimize the multiperiod capital budgeting problem with chance constraints and simple recourse.	Probability that project,'payback' will be obtained at least $\beta\%$ of the time.
[7] Maximize profits of providing heating oil to customers with variable costomer demand and storage contraints.	1. Probability that costomer deman for heating oil will be met with 'high',probability.

		2. Probability that storage capacity is exceeded no more than μ % of the time.
[20]	Minimize cost of diet plan for hospital patients while meeting their nutritional requirements.	Probability that patients nutritional requirements are not met is nom more than α.
[26]	Determine the optimal cropping pattern subject to probabilistic constraints on meeting minimum consumption requirement.	Revenues from cropping pattern selected will not below minimum subsistence more than 100α% of the time.
[30]	Minimize costs of providing staff in a queucing situation.	Probability that demand for a critical resource will not exceed resource capacity more than β% of the time.
[28]	Discussion on incorporation of chance constraints into stochastic planning models for river basins to determine required reservior capacities. Various water yield levels are promised with differing relabilities.	Mode I : maintain a range of discrete storage volumes a stream flows in period + with probability at least P. Mode-II. Probabilities of flow at least a firm and various secondary' flow values in each period and are to be at least

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Chapter - IV

APPLICATION OF STOCHASTIC PROGRAMMING MODEL

CHAPTER - IV

4.1 STOCHASTIC PROGRAMMING APPLIED TO HUMAN RESOURCE PLANNING :

So many normative modelling techniques can be used or many have been used. However the most popular has certainly been linear programming and its extensions linear programming can be used to solve many types of human resource planning problem[14],[35],[38],[48].

In such a model, a set of basic accounting equations are established to describe the relationship between stocks (number of personnel in a given rank and occupation) and flows (transfers, promotions, hirings, releases) which are decision variables. Other constraints can be placed on the stocks and flows the describe, for example labour market restriction on the numbers that can be hired (this may require a supply fore casting model), budgetary limitations. upper and lower bounds on the stocks in various states, the desired ratio of the stock in one state to the stock in another and so on.

An objective function is then specified. For example one may wish to minimize the discounted sum of hiring, training, shortage and salary costs over a number of years Mathematically the constraints may be expressed in the matrix equations.

$$R_t X(t) - S_t X(t-1) = 0 \quad \forall_t \qquad \ldots (4.1.1)$$

$$A_t X(t) = b_t \; X_t \geq 0 \; \forall_t$$

where for a planning period to R_t and S_t are the matrices of the coefficients of accounting constraints, A_t is the matrix of the coefficients of the other constraints, $X(t)$ is a column vector of decision variables (stocks, promotions, transfers, hiring) and b_t is the vector of data concerning bounds on stocks, restrictions on training capacities and transfer flows, ratios between stocks in various states, supply demand, budget, and so on. The objective function can be expressed as

$$\text{Minimize} \sum_t \alpha_t C_t \; X(t) \qquad \ldots (4.1.2)$$

where C_t is a row vector of costs and where α_t is the discount factor for planning period. Goal programming models[12] [42] have been developed to deal with this difficulty. The objective function used in such models weights the importance of various constraints and sub objective according to priorities expressed by one or several decision makers involved. Empirically, the results obtained with these models have been good, despite the lack of any deep theoretical justification for the way in which the objective function weights are obtained, as their determination to a large extent, has been justified by management in terms of the structure and social relationships of the given organization. Mathematically, the programmes, obtained have the form :

$$\min \sum_t \alpha_t \left\{W_t^+ y^+(t) + W_t^- y^-(t)\right\} \qquad \ldots(4.1.3)$$

Subject to

$$R_t X(t) - S_t X(t-1) = 0, \forall_t$$

$$A_t X(t) + I y^+(t) - I y^-(t) = b_t \forall_t$$

$$U_t X(t) = V_t, X(t), y^{\pm}(t) \geq 0 \forall_t$$

where, for a planning period $t, y^+(t)$ and $y^-(t)$ are column vectors of discrepancies from the stated goals, where A_t and b_t contain, respectively the left hand sides and the right hand sides of other constraint that hold in the usual rather than the goal sense, and where W_t^+ and W_t^- are row vectors of the penalty weights associated with the deviations from the fix goals. These weights may be, for example, objectively or subjectively estimated cost.

An Approach To Manpower Planning Under Uncertainty :

Goal programming is appropriate in the case where the vector b_t, which contains information on manpower supply, available budget, demand and so on in program (4.1.3) represents a policy or a state that is fixed and is outside the central of the decision-maker. However, if the resources involved are not imposed or otherwise fixed a priori, if they are necessary to fulfil the role of the organisation, and if their future availability or requirement is net known at the decision point. Whem this is the case, the problem cannot be solved as an ordinary LP because of course, the feasible region is no longer clearly defined. A natural way of tackling this difficulty is to transform model (4.1.3) into dynamic stochastic programming with recourse may be found in Ref.[54] charnes, Cooper, Nichaus, and Sholtz[13].

When the b_t's in programme (4.1.3) are random vectors, the objective function must be changed of one of minimizing the discounted sum of the expected costs (possibly subjective) of deviations from resource availabilities and/or requirements over the entire horizon and the decision variables become functions of the random variables in the model and are more properly called decision, rules. The problem of finding the optimal form of these decision rules cannot in general be solved within the actual state of the art. However, by introducing two assumptions, the model can be reduced to one of finding optimal zero-order decision rules, and such problem are soluble. The variables $\psi^+(t)$ and $\psi^-(t)$ reflect the state of the system at time (t), that is to say, they show whether various resource requirements and/or availabilities (the b_t) have been met, overshot and undershot, When the random vector b_t are eventually observed, the state variable $\psi^+(t)$ and $\psi^-(t)$ are uniquely determined by the values of the $X(t)$ and they do not in themselves effect the values of subsequent decisions(this is reflected in the model by the fact that each of these variable appears in a single constraint). They do, however, provide in period (t). They can therefore be treated as random recourse variables. The only decision rules then required are the X(t).

The management planning system are made for fixed horizon (3-5 years). Planners do not wait to observe all the results of the operation of first year of their programmes though these results will in general, differ from the plan, before planning for the second year, but will determine a programme to cover the entire horizon from outset. They will, of course, revise the initial programme as more information becomes available. In imitation of this procedure, is reasonable to determine the X(t) for all t at the outset ('here and now'), that is to say, to find optimal 'zero order' decision rules.

Under the asumptions, model (4.1.3) reduces to the following multiperiod stochastic programme with simple recourse (SpSR) :

$$\min \ E \ \sum_t Q(X(t), b_t) \qquad \ldots(4.1.4)$$

subject to

$$R_f \ X(t) - S_t \ X\ (t-1) = 0 \quad \forall_t$$

$$U_t \ X(t) = V_t, \ X(t) \geq 0 \quad \forall_t$$

where $Q(X(t), b_t)$ is the second stage programme

$$\min \alpha_t \left\{W_t^+ y^+(t) + W_t^- y^-(t)\right\} \qquad \ldots(4.1.5)$$

Subject to

$$Iy^+(t) - Iy^-(t) = b_t - A_t X(t)$$

$$y^+(t),\ y^-(t) \geq 0$$

From the work of Beale [3], Dantzig(15] and wets [55] on SpSR'S an equivalent deterministic programme can then be obtained. If we denote the marginal probability distribution of the ith element of vector b_t by $F_{it}(h)$ this programme is written as follows :

$$\min \sum_t \sum_i Q_E (Z_i(t), b_{it}) \qquad \ldots(4.1.6)$$

Subject to

$$R_t X(t) - S_t X(t-1) = 0 \qquad V_t \qquad \ldots(4.1.6a)$$

$$A_t X(t) - Z(t) = 0 \qquad V_t \qquad \ldots(4.1.6b)$$

$$U_t X(t) = V_t, X(t) \geq 0 \qquad V_t \qquad \ldots(4.1.6c)$$

where

$$QE(Z_i(t), b_{it}) = \alpha_t(W_{it}^+ W_{it}^-) \, \partial \int_{b_{it} < Z_i(t)} (Z_{i(t)} - h) dF_{it}(h)$$

$$-\alpha_t W_{it}^+ Z_i(t) + \alpha_t W_{it}^+ E(b_{it}) \qquad \ldots(4.1.7)$$

and where $E(b_{it})$ is the expected value of b_{it}.

Note that the objective function of programme(4.1.6) is convex and seperable and that the constraints are all linear.

It should be noted that although model (4.1.6) is presented as an expension of model (4.1.3) as similar programme can be obtained directly from (4.1.1) and (4.1.2). In other words, if one works in terms of real costs instead of discrepancy weights, the approach is equally applicable. The cost of the manpower activities viz

$$\sum_t \alpha_t C_f X(t)$$

can be added directly to the objective function of (4.1.6) without changing the nature of the problem.

4.2 STOCHASTIC REGENERATION MODEL EQUIPMENT REPLACEMENT :

An important example for regeneration model is timber harvesting and replanting problem. Each time the forest is cut the process regenerates itself in the sense that the timber company must have decide how long to wait unit the next harvest period. There is regeneration period occur each time a machine is replaced. Consequently, the decision variables are really the successive intervals between replacement.

The finite horizon deterministic regeneration model can be stated succintly as follows : of the next n periods, starting with the current period, the decision-maker must choose from N alternatives which are indeed $K = 1,2..N$. If alternative K is selected at any period, then the next decision opportunity or regeneration occurs K periods later. When a choice must be made again.(When $n < N$, then the choice is limited to $K = 1,2,..,n$) Assume that the cost of each alternative depends only on K and not on the period at which the alternative is selected, let

$$RK = \begin{pmatrix} \text{Cost of Alternative K valued at the start} \\ \text{of a regeneration period} \end{pmatrix} \quad (4.2.1)$$

An optimal regeneration policy is one that yields the smallest possible total cost over the entire horizon.

$$f_n = \begin{pmatrix} \text{cost of an optimal regeneration policy in which} \\ \text{an alternative must be chosen when n periods remain} \\ \text{until the end of the planning horizon.} \end{pmatrix}$$

...(4.2.2)

Then the values for f_n can be computed by the recursion.

$$f_n = \underset{K=1,2,..N}{\text{minimum}} [R_K + f_{n-k}] \quad f_o = 0, \qquad ...(4.2.3)$$

where $n \geq N$, for $n < N$, the minimum is taken over $K = 1,2,..n$

An important illustration of this model is a simple equipment replacement problem, suppose that the new piece of equipment is purchased at the current period, and suppose that selecting alternative K implies that the new machinery is scrapped and replaced with another piece of equipment. In this content, the regeneration problem is to determine the periods at which to replace the machinery.

For a stochastic version of simple equipment replacement problem, suppose that the machinery may break down before the planned replacement. If, at a regeneration period, the planned replacement decision is K but the machine actually fails during the jth period of usage, then assume that the equipment must be replaced at the start of the subsequent period, let.

K = (planned replacement)
interval

$$p_j = \left(\begin{array}{l}\text{probability that the equipment breaks}\\ \text{down for the first time during } j^{th} \text{ period}\\ \text{of usage.}\end{array}\right)$$

$$r_j = \left(\begin{array}{l}\text{Cost of operating the equipment during}\\ \text{the jth period of usage if the equipment}\\ \text{does not break down)}\end{array}\right)$$

$$p_j + r_j = \left(\begin{array}{l}\text{Cost of operating the equipment if it does}\\ \text{break down during the jth period of usage}\\ \text{when } j < k \text{ where } (S_j > 0)\end{array}\right)$$

and where $\Sigma\, p_j = 1$, r_1 includes the initial purchase cost of the equipment and for expositional simplicity, the equipment is assumed to have no solvage value. We can interpret S_j as a penalty cost for early breakdown.

$$q_K = 1 - \sum_{j=1}^{n} p_j \text{ for } K > 1 \text{ and } q_1 = 1 \qquad \ldots(4.2.4)$$

So that q_K represents the probability that the equipment breaks down for the first time after the (K-1) st period of usage.

Assume that an optimal policy is one that minimizes expected cost over the horizon. If we are at a regeneration period and our planned replacement decisinn is K, then expected cost is comprised of the following. First, we must add the expected operating cost between this and the next (randomly determined) regeneration point. Second, we must include the expected cost at the next regeneration point and beyond in the event that the equipment does not break down before the planned replacement period. And, finally, We must add the expected cost incurred at the regeneration point and beyond in the event that the equipment breaks down before the planned replacement period.

According, the appropriate generalization of (4.1.3) for $n \geq N$ is

$$f_n = \min_{K=1,2,\ldots,N} \left[R_K + f_{n-K}\, q_K + \sum_{j=1}^{K-1} f_{n-j}\, q_j \right]$$

$$fo = 0 \qquad (4.2.5)$$

where now

$$R_K = \sum_{j=1}^{K} r_j q_j + \alpha \sum_{j=1}^{K-1} S_j p_j \qquad \ldots(4.2.6)$$

Ignore the last summation on the right of (4.2.5) and (4.2.6) whenever $K = 1$. For $n < N$, the minimum in (4.2.5) is taken over $K = 1,2,\ldots n$[observe that if a machine never breaks (hence all $p_j = 0$ and every $q_k=1$). Then (4.2.5) reduces to (4.2.3)]. Let $K(n)$ denote an optimal decision found in (4.2.5).

In a situation in which the expected cost of Alternative K depends on the period at which the decision is made, the appropriate expected cost R_{nK} is substituted for R_K in (4.2.5).

To illustrate the calculations, consider the example in Fig. 4.2 where N = 5(we have arbitrary let the probabilities p_2 and p_4 equal 0 in order to reduce the amount of arithmetic required by example). The implied values for R_k are also displayed in Fig. 4.2.

K	p_K	q_K	r_K	S_K	R_K	E(J/K)	$\frac{R_K}{E[j/k]}$
1	1/4	1	100	20	100	1	100
2	0	3/4	$6\frac{2}{3}$	0*	110	$1\frac{3}{4}$	$62\frac{6}{7}$
3	1/4	3/4	20	180	125	$2\frac{1}{2}$	50
4	0	2/4	20	0*	180	3	60
5	0*	2/4	56	0*	208	$3\frac{1}{2}$	$59\frac{3}{7}$

Fig. 4.2

* Each optimal policy is unchanged even if these entries are made positive.

Fig. 4.2: Stochastic replacement Model Example.

For n = 1,(4.2.5) is simply

$$f_1 = R_1 + f_0 q_1 + 0 = 100 + 0\ (1) + 0 = 100 \text{ for } K = 1 \qquad \ldots(4.2.7)$$

So that k(1) = 1. For n = 2 (4.2.5) is

$$f_2 = \min \left\{ \begin{array}{l} R_1+f_1q_1+0 = 100+100\ (1)+0 \\ \qquad\qquad = 200 \text{ for } K = 1 \\ \\ R_2+f_oq_2+f_1p_1 = 110+0(3/4)+100(1/4) \\ \qquad\qquad = 135 \text{ for } K = 2 \end{array} \right\} \quad ..(4.2.8)$$

$$= 135$$

So that $k(2) = 2$. The computations for $n = 3$ and 4 yield $f_3 = 158.75$ with $K(3) = 3$, and $f_4 = 239.69$ with $K(4) = 3$.

For $n = 5$, the recursion (4.2.5) implies.

$$(4.2.9)\quad \alpha_s = \text{minimum} \left\{ \begin{array}{l} R_1 + f_4q_1 + 0 \\ \quad = 100 + 239.69(1)+0 = 339.69 \text{ for } K = 1 \\ R_2 + f_3q_2+f_4p_1 \\ \quad = 110+158.75(3/4)+239.69(1/4) \\ \qquad\qquad = 288.98 \text{ for } K = 2 \\ R_3 + f_2q_3+f_4p_1+f_3p_2 \\ \quad = 125+135.00(3/4)+239.69(1/4)+0 \\ \qquad\qquad = 286.17 \text{ for } K = 3 \\ R_4+f_1q_4+f_4p_1+f_3p_2+f_2p_3 \\ \quad = 180+100.00(2/4) + 239.69(1/4) \\ \quad + 0 + 135(1/4) = 323.67 \text{ for } K = 4 \end{array} \right.$$

$$\left[\begin{array}{l} R_5+f_0q_5+f_4p_1+f_3p_2 \\ \quad + f_2p_3 + f_1p_4 \\ = 208 + 0 + 239.69\ (1/4) \\ \quad + 0 + 135\ (1/4) + 0 \quad = 301.67 \text{ for } K = 5 \end{array}\right]$$

$= 286.17$

So that $K(5) = 3$ continuing in the same fashion we can find that $f_6 = 315.61$ with $K(6) = 3$, and $f_7 = 383.67$ with $K(7) = 3$. The compptation for $n = 7$ with $K = 3$ is (4.2.7).

$$R_3 + f_4q_3 + f_6\, p_1+f_5\, p_2 = 125 + 239.69\ (3/4) + 315.61\ (1/4) + 0 = 386.67 \quad \text{...(4.2.10)}$$

The decision ($K(n) = 3$ is optimal for all $n \geq 3$. Thus if the planning horizon is at least 3 periods, the initial planned replacement decision is for 3 periods, and remains so at every regeneration point until the horizon less than 3 period (when $n = 1$ or 2, the planned replacement is for n period).

4.3 APPLYING STOCHASTIC ALGORITHMS TO A LOCOMATIVE SHEDULING PROBLEM :

The railway networks are faced with the problem of scheduling locomotives, or other forms of traction, so as to cover all the trains in a given timetable at low cost. The variable cost involved are the locomotives themselves and the costs of moving locomotives unproductive from the destination of one train to the starting point of the next (usually known as 'light running'). These are very substantial costs indeed, and so it is well worth taking great care to keep them as low as possible. The scheduling procedure aims to find pattern that can be repeated day after day. However, this does not preclude the occurance of trains which start on one day and end on the next or even later.

Two things need to be determined for each train i:

i) which type of locomative (t_i) should work this train

ii) After this train, which train (r_i) should the locomotive work next.

With t_i and r_i specified for every train i in the timetable, pattern of work, or locomative diagram' can be deduced. The decision as to which physical locomotive should be allocated to a diagram can be left to a later date. Boder[6] devised a method for giving good solutions to small problems, while also allowing the departure times of the train to be flexible, but this method did not produce good results for problems even approaching a realistic size (say, between 100 and 500 trains).

Cost matrix

Let C_{ij} be the cost of 'connecting train' to train j, i.e. of having $j = r_i$. This assumed to be independent type of locomotive involved. Then C_{ij} is defined as $M_{ij} + \mu\ L_{ij}$, where M_{ij} is defined as the number of times midnight passes between the departure of train i and subsequent departure of train j, L_{ij} is the number of minutes taken by a locomotive to travel 'Light' from the destination of train i to the starting location of train j, and and μ are constants.

If there is no type of locomotive compitable with both i and j, then C_{ij} is very large number, effectively infinite.

Mathematical Formulation

Let there be n trains and m types of locomotives. Set p_{iK} = if train i is compitable with type K

= 0 otherwise

Then, every integer i from 1 to n, ti and r_i must be chosen So as to

$$\text{Minimize} \sum_{i=1}^{n} C_i r_i$$

subject to

$$1 \leq r_i \leq n \ (r_i \text{ integer}) \text{ for all } i$$
$$r_j \neq r_i \ (i \neq j)$$

$$1 \leq t \leq m \ (t_i \text{ integer}) \text{ for all } i$$

$$t_i = t_{ri} \text{ for all } i,$$

$$P_i t_i = 1 \text{ for all } i,$$

If there is only locomotive involved, this is a standard linear assignment problem. Conversely, if the locomotive type every train is specified, the problem becomes m independent Linear assignment problems, one for each type.

ALOGRITHMS

Three alogrithms are used to solve this problem. All of them made of assignment algorithms. In every case. Wright's variation of the standard Hungarian method[59] is used to save the time.

The algorithm are as follows :

Algorithms D (deterministic)

Step D_1, solve the problem, ignoring the type constraints and using wright's variation and then change cost matrix accordingly. This gives a cost (LB) which is lower bound for the cost of a feasible solution to the type-constrained problem.

Step D_2. For every type K in turn :

(a) Set $C_{ii} = 0$ for all i with $p_{ik} = 1$

and $p_{i\ell} = 1$ for some $\ell > K$,

(b) Solve the 'partial' assignment problem, i.e. for all with $p_{ik}=1$, find x_i, to minimize cost, subject to the constraint that $i = x_j$ for same j with $p_{jk} = 1$:

(c) for all i with $p_{ik} = 1$ and either $i \neq x_i$ or $p_{i\ell} = 0$ for all $\ell > k$, set $t_i = K$, retain the value of x_i obtained from the assignment algorithm and set $p_{i\ell} = 0$ for all $\ell > k$,

(d) for all i with $p_{ik} = 1$ and $i = x_i$ and $p_{il} = 1$ for some $l > K$, reset C_{ii} to its original value and x_i to zero.

(a), (c) and (d) are unnecessary when $K = m$.

This produce a full set of x_i and t_i, together with a cost which starts at lB, and is then incremented successively by all the steps 2(b) above. The solution procedure is feasible.

This method is used on a railway network.

<u>Algorithm L(Local improvement)</u>

Step L 1 - the same as step D 1

Step L 2 - use random numbers to set t_i for all i,

subject to $pit_i = 1$

Step L 3 - Solve the assignment problem subject to $t_i = tr_i$ for all i(this requires a very minor changes to the assignment algorithm used). This produces a feasible solu, with a cost FC.

Step L 4 - For each train i in turn, the following for each type K in turn such that $t_i \neq K$ and $p_{ik} = 1$:

(a) Set $t_i = K$, $r_i = 0$ and for the train S with $r_s = i$, set $r_s = 0$

(b) From this position, we can find a new assignment solution subject to $t_j = tr_i$ for all j (but about the assignment algorithm as soon as the cost > FC, if it ever does).

(c) If the new cost $\geq$ FC, reset t_i and r_j to their previous values (NB the cost matrix does not need to be reset, because Wright's variation does not change the cost matrix).

(d) If the new cost < FC, keep the new solution, reset the matrix by carrying out the row and column subtractions indicated by Wright's variation, and set equal to the new cost.

This step is continued until a point is reached where no solu. change has taken place since the last the pair (i/k) is investigated ie. a local optimum has been reached.

Algorithm A (Simulated annealing)

Step A 1 - as Step D 1

Step A 2 - as step L 1

Step A 3 - as step L 3

Step A 4 - the three parameter, N, T_o and T_N

Seting $T = T_o$ we are using the formula for calculating parameter β.

$$\beta = \frac{1}{N-1} \left(\frac{1}{T_o} - \frac{1}{T_N} \right)$$

Step A 5 - the following N times :

(a) Find a random number R from the uniform distribution between o and 1. MAX = FC - T. Loge (R).

(b) Select at random a train i and type K such that $t_i \neq K$ and $p_{ik} = 1$, setting $f_i = K$, $r_i = 0$, for the train S with $r_s = i$, set $r_s = 0$.

(c) From this position, we find a new assignment solution subject to $t_j = tr_j$ for all$_j$ (but abort the assignment algorithm as soon as the cost $\geq$ MAX, if it ever does).

(d) It the new cost $\geq$ MAX, reset t_i and r_j to their previous values (NB the cost matrix does not need to be reset, because WWright's variation does not change the cost matrix).

(e) If the new cost < MAX, then the new solution and reset the matrix by carrying out the row and column subtraction indicated by Wright's variation. Now we set FC is equal to the new cost.

(f) If the cost is the lowest achieved during this step, we will make a record of it.

(g) Set $T = T/(1 + \beta T)$.

Step a 6 : Carry out L4, first from the current solution, then from the best solution reached in step A_5. Accept the cheaper of the two solutions reached.

Because of the random element introduced in steps A_2 and A_5, steps A_2 to A_6 many be repeated as many times as required, and only cheapest solution ever ~~x~~ found need be kept. This can be continue until a solution is reached which is deemed to satisfactory, or no more time is available.

This algorithm comes under the heading of 'simulated annealing', using the varition propounded by Lundy and Mees [29].

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4.4 A STOCHASTIC CONSTRAINED OPTIMIZATION MODEL FOR DETERMINING COMMERCIAL FISHING

A stochastic constrained optimization model developed for multi-species fishery that sets the seasonal catch by species, by geographical area, and by month of the season. Here a problem has been developed on this model. The problem has 228 variable and 630 constraints which are summarized in Table 1. The model and simulation procedure for its solution are described here in more detail.

Decision Variables :

The decision variables are X_{ijk}, the numbers of tons of species (i) to be caught from geographical area (j) in time period (K). There are three species - prawns. Lobsters and cephalopods fish, and either 5 or 9 geographical areas per species. Prowns fish has 9 areas, lobsters and cephalopods fishes each have 5 areas. These areas are taken at southwest of India coastal water by the central Marine Fisheries Research Institute (CMFRI) Cochin. There are 12 time periods one for each month of the fishing season. This one year planning horizon is supported by both annual biological forecast data and annual negotiated contracts for species prices from mechanise boats to processors.

Disaggregating the decision variables by species(3) geographical area (5-9) and month(12) results in a model with 228 individual decision variables.

TABLE – 1

Configuration of the Stocnastic Programming Problem

Problem Element	Description	Number of Elements
Decision	X_{ijk}–the target number of tons of species to be caught in each geographical area in each month.	228
Objective function	To maximize the contribution of the mechanise boat for all species and areas over one year planning horizon.	1
	Boat capacities – Maximum catching capacities of each species in each time total maximum catching capacity for all species in each time period, and total catching capacity for all species across all time periods.	49
	Weather – Maximum number of fishing days in each area in each month for each species	228
Constraints	Biological Considerations – Each species in each geographical area has time periods when fishing must be oviodeel,	54
	Markets – Maximum amount of each species which can be sold in each time period.	36

Processing - Maximum processing capacity for each month and region to each species and over all species.	196
Recruitment - The maximum amount of each area which can be caught over the entire season.	19
Boat Utilization - leveling catching capacity across months for each species and over all species.	48

Objective Functions

The objective function for the model is

$$\text{Max } Z = \sum_{i=1}^{3} \sum_{j=1}^{J_1} \sum_{k=1}^{12} C_{ijk} X_{ijk}$$

Which maximizes the industry contribution (Profits) of the mechanise boat for fishing C_{ijk} is the contribution per ton for each species (i) in each geographical area (j) and each month (K).

C_{ijk} includes such variable cost factor as labour, fuel, repairs and maintenance, and supplies prices are negotiated for the entire year but these prices hence seasonal patterns and are adjusted for yields, seasonal price patterns are necessary because processors expect to seasonally adjust their prices to wholesalers based on fore casted supply and demand fluctuations yield adjustments are necessary because x immediately after a fish losses its shell (molts), if meat weight relative to its total weight climbs until its new shell is entirely filled.

Constraints

$$\sum_{i=1}^{3} \sum_{j=1}^{j_i} \sum_{k=1}^{12} A_{ijk}X_{ijk} <, =, \geq Bc$$

$$C = 1,2, \ldots 630$$

A_{ijk} are the technological requirements (substitution rates) for each unit of X_{ijk} and b_i is the requirement for constraint. There are seven sets of constraints of the model. Those concerning boat capacities, weather, biological considerations, markets, processing, recruitment and boat utilization. All these thing are shown in table 1.

Probabilistic Elements

Four major sources of variation are x treated explicity in the model. The catch per species, catch per unit of effort (CPUE), whether, and market factors. These uncertainties inflict randomness upon certain of the C_{ijk}, A_{ijk}, and b_c elements. Each of the sources of variation, their causes, and their effects are summarized in Table 2.

Some probabilistic elements of the model, for example the CPUE, were represented by discrete empirical distributions when historical data were appropriate for the assumption of the model. In other cases for example to model recruitment, probabilistic elements were represented by normal distributions development from forecasts or other estimates.

Table – 2

Probabilistic Elements of the Model

Source of uncertainty	Causes of uncertainty	Principal Elements of Model which are Affected.
1. Recruitment or catch per species limit.	Imprecise cutoff species catch at processors, tons to days conversions and biological factors.	b_c of each species-species, area constraint(affects 19 constraints).
2. Efficiency of mechanise boats catch per unit of effert (CPUE)	Dispersion of species weather, abundance of species, type of boats employed.	A_{ijk} of the boat capacity constraints (affects 49 constraints) C_{ijk}(affects all 228 coefficients)
3. Weather	Wind, frigid temperature and high tides reduce catch per unit of effort (CPUE) and number of fishing days.	A_{ijk} of the boat capacity constraints (affect 49 constraint

		b_c of each area time period weather constraint (affects 228 constraints)
		A_{ijk} of each area time period weather constrained (affects 228 constraints)
		C_{ijk}(affects all 228 coefficients)
4. Markets	Factors affecting quantity of each species which will be purchased.	b_c of each species time period markets constraint(affect 36 constraints).

The Simulation

Here many of the C_{ijk}, A_{ijk}, and b_c of the linear programming problem may be random variables, a procedure called Experimental solution of stoch. Programming (ESSp) is used to develop probabilistic values of objective function, decision variables, and stock variables. This procedure was pioneered by Sengupta and Tinther[45] and later independently developed and applied by this author [19]. Fig. 4.4 shows that ESSp procedure selects a single set of parameter values for the problem parameter may be either consants or values randomly selected from their discrete or continous distribution), executes the mathematical programming algorithm, and stores that cycle's results. This sequence of steps is repeated until the desired number of cycles have been completed. The desired number of cycles x can be determined either by specifying in advance of execution of the model a specific number of cycles or frackling algorithm can be employed which heuristically terminates the program.

A statistical summary of the resulting values of the objective function decision variables, and stock variables is then developed. The chief advantage of this procedure compared to those that assume constant parameters or use expected values for the parameters is that the dispersion of the objective function can be obtained. Thus, confidence intervals can be stated for the objective function. In the ESSP procedure. The mean of the cycle objective function values represents the expected industry contribution if the mean values of the decision variables were adopted as the decision set. The standard deviation of the cycle objective function values is a measure of the confidence that decision, makers can place in the expected industry.

P.T.O

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Flow Chart

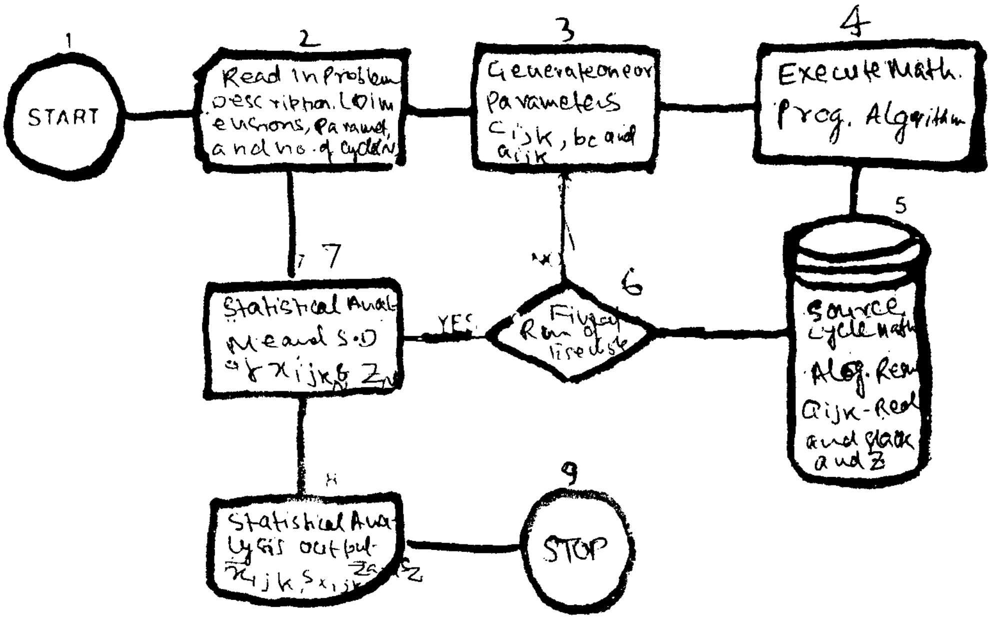

Fig 4.4 ESSP simulation procedure

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